Chapter 4

Lagrangian submanifolds and generating functions

Motivated by theorem 3.9 we will now study properties of the manifold $\Lambda_\phi \subset X \times (\mathbb{R}^n \setminus \{0\})$ for a clean phase function ϕ. As shown in section 3.3 Λ_ϕ is an immersed submanifold, and since ϕ is positive homogeneous of degree 1 Λ_ϕ is conic (i.e. if $(x, \eta) \in \Lambda_\phi$, then $(x, \lambda\eta) \in \Lambda_\phi$ for all $\lambda \in \mathbb{R}^+$). A more important property is that Λ_ϕ is a Lagrangian submanifold. If we consider $X \times (\mathbb{R}^n \setminus \{0\})$ to be $T^*X \setminus \{0\}$ and let σ be the canonical symplectic form,

$$\sigma = \mathrm{d}\eta_k \wedge \mathrm{d}x^k,$$

recall then that a submanifold $M \subset X \times (\mathbb{R}^n \setminus \{0\})$ is called Lagrangian if for any $(x, \eta) \in M$ and $w \in T_{(x,\eta)} X \times (\mathbb{R}^n \setminus \{0\})$, $w \in T_{(x,\eta)} M$ if and only if

$$\sigma(w, v) = 0, \quad \text{for all } v \in T_{(x,\eta)} M.$$

Another necessary and sufficient condition for M to be Lagrangian is that $\sigma|_{TM} = 0$ and $\dim(M) = n$. Conic Lagrangian manifolds have the additional property that the canonical one form, $\alpha = \eta_j \mathrm{d}x^j$, vanishes on TM.

Theorem 4.1. *If ϕ is a clean phase function then $\Lambda_\phi \subset X \times (\mathbb{R}^n \setminus \{0\})$ is an immersed conic Lagrangian submanifold.*

Proof. By the calculations in section 3.3

$$T\Lambda_\phi = \left\{ a^j \partial_{x^j} + (a^j \partial^2_{x^k x^j} \phi + b^j \partial^2_{x^k \xi^j} \phi) \partial_{\eta^k} \ : \ a^j \partial^2_{\xi^k x^j} \phi + b^j \partial^2_{\xi^k \xi^j} \phi = 0 \right\}.$$

Thus for any vectors v, and $w \in T\Lambda$

$$v = a^j \partial_{x^j} + (a^j \partial^2_{x^k x^j} \phi + b^j \partial^2_{x^k \xi^j} \phi) \partial_{\eta^k}, \quad w = \tilde{a}^j \partial_{x^j} + (\tilde{a}^j \partial^2_{x^k x^j} \phi + \tilde{b}^j \partial^2_{x^k \xi^j} \phi) \partial_{\eta^k},$$

we have

$$\begin{aligned} 2\sigma(v, w) &= (a^j \partial^2_{x^k x^j} \phi + b^j \partial^2_{x^k \xi^j} \phi) \tilde{a}^k - (\tilde{a}^j \partial^2_{x^k x^j} \phi + \tilde{b}^j \partial^2_{x^k \xi^j} \phi) a^k \\ &= b^j \tilde{a}^k \partial^2_{x^k \xi^j} \phi - \tilde{b}^j a^k \partial^2_{x^k \xi^j} \phi \\ &= -b^j \tilde{b}^k \partial^2_{\xi^k \xi^j} \phi + \tilde{b}^j b^k \partial^2_{\xi^k \xi^j} \phi = 0. \end{aligned}$$

This shows that Λ_ϕ is Lagrangian and thus completes the proof. $\square$

When ϕ is a clean phase function then we say that the conic Lagrangian manifold Λ_ϕ is parametrized by ϕ. We might then ask whether a given conic Lagrangian manifold can be parametrized by a clean, or perhaps nondegenerate, phase function. The answer to this question, which we will address in the next section, is yes.

4.1 Coordinates and construction of generating functions

In this section we will show how, given an arbitrary immersed Lagrangian manifold $\Lambda \subset X \times \mathbb{R}^n \setminus \{0\}$ and a point $(x_0, \eta_0) \in \Lambda$, we can find a nondegenerate phase function ϕ such that in a neighborhood of (x_0, η_0) $\Lambda = \Lambda_\phi$. To accomplish this goal we begin by defining some coordinates on Λ in a neighborhood of (x_0, η_0) as in the next lemma.

Lemma 4.2. *Let $\Lambda \subset X \times \mathbb{R}^n \setminus \{0\}$ be an immersed Lagrangian submanifold and $(x_0, \eta_0) \in \Lambda$. Then there exists a partitioning, I and J, of the indices $\{1, \dots, n\}$ (i.e. $I \cup J = \{1, \dots, n\}$ and $I \cap J = \emptyset$) such that the map $\Lambda \mapsto (x_{I_1}, \dots, x_{I_{|I|}}, \eta_{J_1}, \dots, \eta_{J_{|J|}})$ is a diffeomorphism on some neighborhood of (x_0, η_0)*

Proof. First we show that there exists a partition I' and J' as described in the theorem such that

$$V = \operatorname{span}\left\{\partial_{x_{I'_1}}, \dots, \partial_{x_{I'_{|I'|}}}, \partial_{\eta_{J'_1}}, \dots, \partial_{\eta_{J'_{|J'|}}}\right\}$$

is transverse to $T_{(x_0,\eta_0)}\Lambda$. To do this, we begin by taking I' to be a maximal subset of $\{1, \dots, n\}$ such that $V' = \operatorname{span}\{\partial_{x_{I'_1}}, \dots, \partial_{x_{I'_{|I'|}}}\}$ is transverse to $T_{(x_0,\eta_0)}\Lambda$. If $J' = \{1, \dots, n\} \setminus I'$, then this implies that if we take any coefficients b^j not all zero there exists a $\lambda \neq 0$ and some coefficients $\tilde{a}^i$ such that

$$\tilde{a}^i \partial_{x_{I'_i}} + \lambda b^j \partial_{x_{J'_j}} \in T_{(x_0,\eta_0)}\Lambda.$$

Then for any coefficients a^i

$$\sigma(\tilde{a}^i \partial_{x_{I'_i}} + \lambda b^j \partial_{x_{J'_j}}, a^i \partial_{x_{I'_i}} + b^j \partial_{\eta_{J'_j}}) = -\lambda |b^j|^2 \neq 0$$

which implies that $a^i \partial_{x_{I'_i}} + b^j \partial_{\eta_{J'_j}} \notin T_{(x_0,\eta_0)}\Lambda$ since Λ is Lagrangian. This proves the initial claim that V is transverse to $T_{(x_0,\eta_0)}\Lambda$.

Now let us take an arbitrary coordinate map ψ for Λ defined on a neighborhood U of (x_0, η_0). Then Λ is defined locally in a neighborhood of (x_0, η_0) by

$$\psi^{-1}(y) - (x, \eta) = 0. \tag{4.1}$$

The range of $D\psi^{-1}|_{\psi(x_0,\eta_0)}$ is equal to $T_{(x_0,\eta_0)}\Lambda$ and so the fact that V is transverse to $T_{(x_0,\eta_0)}\Lambda$ implies that, by the implicit function theorem, (4.1) locally defines y and

$(x_{I'_1}, \ldots, x_{I'_{|I'|}}, \eta_{J'_1}, \ldots, \eta_{J'_{|J'|}})$ as a function of $(x_{J'_1}, \ldots, x_{J'_{|J'|}}, \eta_{I'_1}, \ldots, \eta_{I'_{|I'|}})$. With $I = J'$ and $J = I'$ this proves the result. $\square$

We will make quite a bit of use of these types of coordinates on Lagrangian manfolds, and to save notation in the sequel we simply write (x^I, η_J) for $(x_{I_1}, \ldots, x_{I_{|I|}}, \eta_{J_1}, \ldots, \eta_{J_{|J|}})$, and the corresponding notation for vectors as well. Additionally we will use the Einstein summation convention over the subsets specified by I or J so for example

$$\eta_J \mathrm{d}x^J := \sum_{j=1}^{|J|} \eta_{J_j} \mathrm{d}x^{J_j}.$$

Now we use the coordinates given by the previous lemma to prove the existence of a nondegenerate phase function parametrizing Λ.

Theorem 4.3. *Let $\Lambda \subset X \times \mathbb{R}^n \setminus \{0\}$ be an immersed conic Lagrangian submanifold, $(x_0, \eta_0) \in \Lambda$, and I, J be the partitioning given by lemma 4.2. Then there exists a smooth function $S(x_I, \eta_J)$ defined in a conic neighborhood of $((x_0)_I, (\eta_0)_J)$ that is homogeneous of degree 1 with respect to η_J and such that*

$$\partial_{x^I} S = -\eta_I(x^I, \eta_J), \quad \partial_{\eta_J} S = x^J(x^I, \eta_J)$$

where locally

$$\Lambda = \{(x^I, x^J(x^I, \eta_J), \eta_I(x^I, \eta_J), \eta_J)\}.$$

In this case

$$\phi(x, \eta_J) = -S(x^I, \eta_J) + x^J \eta_J$$

is a nondegenerate phase function that parametrizes Λ.

Proof. This proof basically follows from the fact that the canonical 1-form vanishes on $T\Lambda$ since Λ is a conic Lagrangian manifold. Indeed, recall that the canonical one form is

$$\eta_j \mathrm{d}x^j$$

and so the fact that this vanishes on $T\Lambda$ implies

$$\eta_{J'} \left(\frac{\partial x^{J'}}{\partial x^{\tilde{I}}}(x^I, \eta_J) \mathrm{d}x^{\tilde{I}} + \frac{\partial x^{J'}}{\partial \eta_{\tilde{J}}}(x^I, \eta_J) \mathrm{d}\eta_{\tilde{J}} \right) = -\eta_{\tilde{I}}(x^I, \eta_J) \mathrm{d}x^{\tilde{I}}. \tag{4.2}$$

Now we take

$$S(x^I, \eta_J) = \eta_{\tilde{J}}\, x^{\tilde{J}}(x^I, \eta_J) \tag{4.3}$$

and we may check using (4.2) that this function satisfies the required identities. To check that ϕ as defined in the theorem is a nondegenerate phase function note that

$$\partial_{\eta_{\tilde{J}}} \phi(x^I, x^J, \eta_J) = -x^{\tilde{J}}(x^I, \eta_J) + x^{\tilde{J}}$$

and so

$$C_\phi = \{(x^I, x^J(x^I, \eta_J), \eta_J)\}$$

is a smooth manifold of dimension n. Finally, we have

$$T_\phi(C_\phi) = (x^I, x^J(x^I, \eta_J), \eta_I(x^I, \eta_J), \eta_J) = \Lambda$$

which completes the proof.

□

In the context of theorem 4.3 S is called a generating function for Λ. In fact the explicit construction of S given by (4.3) will be quite important.

4.2 Canonical transformations and their graphs

We now look more specifically at Lagrangian manifolds defined over product spaces. In particular we consider the case when $X = Y' \times X'$ where $Y' \subset \mathbb{R}^m$ and $X' \subset \mathbb{R}^m$ are open. It will be important when we move on to consider Fourier integral operators to understand Lagrangian manifolds on $X \times (\mathbb{R}^{2m} \setminus \{0\})$ that are given as the graph of a map from $X' \times (\mathbb{R}^m \setminus \{0\})$ to $Y' \times ((\mathbb{R}^m \setminus \{0\})$. We will use the notation (x, ξ) and (y, η) respectively for coordinates on the spaces $X' \times (\mathbb{R}^m \setminus \{0\})$ and $Y' \times ((\mathbb{R}^m \setminus \{0\})$. These spaces then respectively carry canonical symplectic forms $\sigma_{X'}$ and $\sigma_{Y'}$ given by

$$\sigma_{X'} = \mathrm{d}\xi_j \wedge \mathrm{d}x^j, \quad \text{and} \quad \sigma_{Y'} = \mathrm{d}\eta_j \wedge \mathrm{d}y^j.$$

The canonical symplectic form σ on $X \times (\mathbb{R}^{2m} \setminus \{0\})$ is given by

$$\sigma = \sigma_{Y'} + \sigma_{X'}.$$

Definition 4.4 (Canonical transformation/twisted canonical transformation). *A mapping $\chi : X' \times ((\mathbb{R}^m \setminus \{0\}) \to Y' \times (\mathbb{R}^m \setminus \{0\})$ is called a twisted canonical transformation or a twisted symplectomorphism if it is a diffeomorphism and*

$$\sigma_{X'} = -\chi^* \sigma_{Y'}. \tag{4.4}$$

χ is a local twisted canonical transformation if (4.4) holds but χ is only a local diffeomorphism. If (4.4) holds without the negative sign then χ is called either a canonical transformation or a symplectomorphism.

As shown in the next theorem, the graph of a twisted canonical transformation is a Lagrangian submanifold of $X \times (\mathbb{R}^{2m} \setminus \{0\})$.

Theorem 4.5. *Suppose that χ is a twisted canonical transformation from $X' \times (\mathbb{R}^m \setminus \{0\})$ to $Y' \times ((\mathbb{R}^m \setminus \{0\})$ and let π_y and π_η denote the projection from $Y' \times (\mathbb{R}^m \setminus \{0\})$ onto either Y' or $\mathbb{R}^m \setminus \{0\}$. Then*

$$\Lambda = (\pi_y \circ \chi(x, \xi), x, \pi_\eta \circ \chi(x, \xi), \xi) \subset (Y' \times X') \times (\mathbb{R}^{2m} \setminus \{0\})$$

is a embedded Lagrangian submanifold. If the mapping χ is positive homogeneous of degree 1 in the fibers (i.e. if $\chi(x,\lambda\xi) = (\pi_y \circ \chi(x,\xi), \lambda\pi_\eta \circ \chi(x,\xi))$ for $\lambda \in \mathbb{R}^+$), then Λ is also conic.

Proof. Since Λ is the graph of a diffeomorphism it is an embedded submanifold. In fact $(x,\xi) \in X' \times (\mathbb{R}^m \setminus \{0\})$ provide global coordinates for Λ. For any point $z \in \Lambda$ and corresponding $(x,\xi) \in X' \times (\mathbb{R}^m \setminus \{0\})$

$$T_z\Lambda = D\chi(T_{(x,\xi)}(X' \times (\mathbb{R}^m \setminus \{0\}))) \times T_{(x,\xi)}(X' \times (\mathbb{R}^m \setminus \{0\})).$$

Thus any vector in $T_z\Lambda$ may be written as $(D\chi(v), v)$ for some $v \in T_{(x,\xi)}(X' \times (\mathbb{R}^m \setminus \{0\}))$. Let $(D\chi(v), v)$ and $(D\chi(w), w) \in T_z\Lambda$. Then using (4.4)

$$\begin{aligned} \sigma((D\chi(v), v), (D\chi(w), w)) &= \sigma_{X'}(v,x) + \sigma_{Y'}(D\chi(v), D\chi(w)) \\ &= \sigma_{X'}(v,x) + \chi^*\sigma_{Y'}(v,w) \\ &= \sigma_{X'}(v,x) - \sigma_{X'}(v,w) = 0. \end{aligned}$$

This shows that Λ is Lagrangian. The last statement about homogeneous mappings in the fibers is trivial. □

Now we would like to apply the results of section 4.1 to the case considered in theorem 4.5. We begin by proving the following variant of lemma lem:Lambdacoord.

Lemma 4.6. *Suppose that $\chi : X' \times (\mathbb{R}^m \setminus \{0\}) \to Y' \times ((\mathbb{R}^m \setminus \{0\})$ is a twisted canonical transformation and Λ is the graph of χ. If $(y_0, x_0, \eta_0, \xi_0) \in \Lambda$ then there exists a partitioning I,J of $\{1, \dots n\}$ such that $\Lambda \mapsto (y, x^I, \xi_J)$ is a diffeomorphism on some neighborhood V of $(y_0, x_0, \eta_0, \xi_0)$. If Λ is conic, then we can take V to be conic as well.*

Proof. First note that since χ is a twisted canonical map so is χ^{-1}. Thus, since $\Lambda_{y_0} = \{(y_0, \eta) \ : \ \eta \in \mathbb{R}^n \setminus \{0\}\}$ is a Lagrangian submanifold of $Y' \times (\mathbb{R}^m \setminus \{0\})$, $\chi^{-1}(\Lambda_{y_0})$ is a Lagrangian submanifold of $X' \times (\mathbb{R}^m \setminus \{0\})$. Therefore, following the beginning of the proof of lemma 4.2, we can find a partition I', J' such that

$$\mathrm{span}\{\partial_{x^{I'}}, \partial_{\xi_{J'}}\}$$

is transverse to $T_{(x_0,\xi_0)}(\chi^{-1}(\Lambda_{y_0}))$. Since χ is a diffeomorphism this implies that

$$D\chi(T_{(x_0,\xi_0)}(\chi^{-1}(\Lambda_{y_0})))$$

is transverse to

$$T_{(y_0,\eta_0)}\Lambda_{y_0} = \mathrm{span}\left\{\partial_{\eta_j} \ : \ j = 1, \dots, m\right\}.$$

Now, note that Λ is defined by

$$(y,\eta) - \chi(x,\xi) = 0.$$

The above considerations imply by the implicit function theorem that in a neighborhood of $(y_0, x_0, \eta_0, \xi_0)$ this relation defines η, $x^{I'}$, and $\xi_{J'}$ as smooth functions of the other variables, and thus proves the result except the statement about conic Λ.

To prove the statement about conic Λ, let V be a neighborhood of $(y_0, x_0, \eta_0, \xi_0)$ in Λ, which we have already shown exists, on which the map $\Psi : \Lambda \ni (y, x, \eta, \xi) \mapsto (y, x^I, \xi_J)$ is a diffeomorphism. Note first that since Λ is conic $\{\xi_J = 0\} \cap V = \emptyset\}$. Now let

$$V' = \left\{(y, x, \lambda\eta, \lambda\xi) \ : \ (y, x, \eta, \xi) \in V, \quad \lambda \in \mathbb{R}^+\right\}$$

be the smallest conic set containing V. First, it is not difficult to see that for any $\lambda \in \mathbb{R}^+$, the map $(y, x, \eta, \xi) \mapsto (y, x, \lambda\eta, \lambda\xi)$ is a diffeomorphism, and therefore Ψ is a diffeomorphism on λV. Thus Ψ is a local diffeomorphism on V' and all that remains is to show that Ψ is injective on V'.

Now we can also assume without loss of generality that Ψ is a diffeomorphism on a neighborhood V (we are renaming V) of the set

$$W = \{(y, x, \eta/|\xi_J|, \xi/|\xi_J|) \ : \ (y, x, \eta, \xi) \in V'\}$$

which we may also assume is path connected (recall that $\xi_J \neq 0$ on V and therefore on V' as well). Suppose $\Psi(y_1, x_1, \eta_1, \xi_1) = \Psi(y_2, \xi_2, \eta_2, \xi_2)$ for $(y_i, x_i, \eta_i, \xi_i) \in V'$ so that

$$\Psi(y_1, x_1, \eta_1/|(\xi_1)_J|, \xi_1/|(\xi_1)_J|) = \frac{|(\xi_2)_J|}{|(\xi_1)_J|} \cdot \Psi(y_2, x_2, \eta_2/|(\xi_2)_J|, \xi_2/|(\xi_2)_J|)$$

where the "$\cdot$" indicates multiplication only in the ξ_J factor. If $|(\xi_1)_J| = |(\xi_2)_J|$ then we are done since we know that Ψ is injective on W. Otherwise we know that $y_1 = y_2$, $(x_1)^I = (x_2)^I$, and $(\xi_1)_J/|(\xi_1)_J| = \frac{|(\xi_2)_J|}{|(\xi_1)_J|}(\xi_2)_J/|(\xi_2)_J|$. Now take a path $\xi(t)$ from $(\xi_1)_J/|(\xi_1)_J|$ to $(\xi_2)_J/|(\xi_1)_J|$ contained in $\mathbb{S}^{|J|-1}$ such that $(y_1, (x_1)_J, \xi(t))$ is in the image of $\Psi(W)$. Then since Ψ is a diffeomorphism on a neighborhood of W this path parametrizes a path from $(y_1, x_1, \eta_1/|(\xi_1)_J|, \xi_1/|(\xi_1)_J|)$ to $(y_2, x_2, \eta_2/|(\xi_2)_J|, \xi_2/|(\xi_2)_J|)$ contained in V'. The derivative with respect to t of Ψ applied to this path points in the radial direction with respect to ξ_J, but this contradicts the fact that Ψ is a diffeomorphism in a neighborhood of W since the differential of Ψ also maps the radial direction with respect to (η, ξ) to the radial direction with respect to ξ_J, and thus proves the result. □

The main point of the previous lemma is that when Λ is the graph of a canonical transformation we can always in a neighborhood of any point take as coordinates (y, x^I, ξ_J). We can also apply theorem 4.3 and its proof to show that when Λ is defined by

$$\eta = \eta(y, x^I, \xi_J), \quad x^J = x^J(y, x^I, \xi_J), \quad \text{and} \quad \xi_I = \xi_I(y, x^I, \xi_J)$$

the function

$$S(y, x^I, \xi_J) = \xi_{\tilde{J}} x^{\tilde{J}}(y, x^I, \xi_J)$$

satisfies

$$\partial_y S = -\eta(y, x^I, \xi_J), \quad \partial_{x^I} S = -\xi_I(y, x^I, \xi_J), \quad \text{and} \quad \partial_{\xi_J} S = x^J(y, x^I, \xi_J)$$

(i.e. S is a generating function for Λ) and

$$\phi(y, x^I, x^J, \xi_J) = -S(y, x^I, \xi_J) + x^J \xi_J$$

is a nondegenerate phase function that parametrizes Λ.

Next we will look at the particular case when χ is given by the flow of Hamiltonian vector field.

4.2.1 An important example: The flow of a Hamiltonian vector field

Let $H(x,\xi) \in C^\infty(X' \times (\mathbb{R}^m \setminus \{0\})$ be positive homogeneous in ξ of degree 1. We will refer to H as the Hamiltonian. Associated to H through the symplectic form $\sigma_{X'}$ is a so-called Hamiltonian vector field defined by

$$V_H = \partial_{\xi_j} H(x,\xi)\partial_{x^j} - \partial_{x^j} H(x,\xi)\partial_{\xi_j}.$$

For given (x,ξ) let $(y(t,x,\xi), \eta(t,x,\xi))$ be the solution of system of ODEs

$$\partial_t y^j(t,x,\xi) = \partial_{\xi_j} H(y(t,x,\xi),\eta(t,x,\xi)), \quad \partial_t \eta_j(t,x,\xi) = -\partial_{x^j} H(y(t,x,\xi),\eta(t,x,\xi))$$

with initial conditions $(y(0,x,\xi),\eta(0,x,\xi)) = (x,\xi)$. The flow of the Hamiltonian vector field associated to H is the mapping

$$\chi_t(x,\xi) = (y(t,x,\xi),\eta(t,x,\xi)).$$

Since H is homogeneous in ξ of degree α the mapping χ_t is homogeneous of degree 1. Therefore the domain of this map is a conic open subset of $\mathbb{R}_t \times X'_x \times (\mathbb{R} \setminus \{0\})_\xi$. We prove the following theorem showing essentially that for every t, χ_t is a local canonical transformation that also preserves the Hamiltonian.

Theorem 4.7. *Suppose that U is an open conic set on which χ_t is defined for some t. Then $\chi_t|_U$ is a canonical transformation onto its image. Also, $H(\chi_t(x,\xi))$ is constant with respect to t.*

Proof. For the statement that $H(\chi_t(x,\xi))$ is constant in t we simply calculate

$$\begin{aligned}\partial_t H(\chi_t(x,\xi)) &= \partial_{x^j} H(\chi_t(x,\xi))\partial_t y^j(t,x,\xi) + \partial_{\xi_j} H(\chi_t(x,\xi))\partial_t \eta_j(t,x,\xi)\\ &= \partial_{x^j} H(\chi_t(x,\xi))\partial_{\xi_j} H(\chi_t(x,\xi)) - \partial_{\xi_j} H(\chi_t(x,\xi))\partial_{x^j} H(\chi_t(x,\xi))\\ &= 0.\end{aligned}$$

The other point follows from Cartan's formula [**?**] which says

$$\mathcal{L}_{V_H}\sigma_{X'} = \iota_{V_H}(\mathrm{d}\sigma_{X'}) + \mathrm{d}(\iota_{V_H}(\sigma_{X'})).$$

Here $\mathcal{L}_{V_H}$ is the Lie derivative and ι_{V_H} is the interior multiplication by V_H. The symplectic form is closed, and so the first term above is zero. On the other hand, from the definition $\iota_{V_H}(\sigma_{X'}) = \mathrm{d}H$, and so since $\mathrm{d}\,\mathrm{d}H = 0$ the second term vanishes as well. Thus the Lie derivative with respect V_H of the symplectic form is zero, and so $\chi_t^*\sigma_{X'}$ is constant with respect to t. Since $\chi_0(x,\xi) = (x,\xi)$ this implies the result. □

This theorem implies in particular that

$$\tilde{\chi}_t(x,\xi) = \chi_t(x,-\xi)$$

is a twisted canonical transformation. The corresponding conic Lagrangian manifold is

$$\Lambda = \{(y,x,\eta,\xi) \ : \ (x,\xi) \in U, \quad y = y(t,x,-\xi), \quad \eta = \eta(t,x,-\xi)\}$$

where U is the open conic set from theorem 4.7.

4.3 Half densities

In order to study coordinate invariant properties of oscillatory integrals and Fourier integral operators it is necessary to introduce the concept of half densities. Over a given manifold M, half densities are sections of a complex line bundle over M whose transition maps are given by the absolute value of the Jacobian determinant of the change of coordinates maps raised to the power $1/2$. We now make this precise mostly following the presentation of [**?**].

Suppose that M is an n-dimensional manifold. For every $x \in M$ let $\Lambda^n T_x M$ be the dual space of the space of $\Lambda^n_x M$ which are the n-linear alternating forms on $T_x M$. Since $\Lambda^n_x M$ is 1 dimensional so is $\Lambda^n T_x M$. Now for $\alpha \in \mathbb{R}$ we define $\Omega^\alpha_x(M)$ to be the space of maps

$$\rho : \Lambda^n T_x M \setminus \{0\} \to \mathbb{C}$$

such that $\rho(\lambda v) = |\lambda|^\alpha \rho(v)$ for all $v \in \Lambda^n T_x M \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$.

We now construct a complex line bundle over M given by

$$\Omega^\alpha(M) = \bigcup_{x \in M} \Omega^\alpha_x M$$

whose fibers over each point $x \in M$ are $\Omega^\alpha_x(M)$. We will refer to $\Omega^\alpha(M)$ as the α-density bundle although we actually only ever consider the cases $\alpha = 1$, which corresponds with densities, and $\alpha = 1/2$, which corresponds with half densities. First we describe the local trivializations. Suppose that $\Psi : U \subset M \to U' \subset \mathbb{R}^n$ is a coordinate map. Then at a given point $x \in U$ with coordinate representation $(x^1, \dots, x^n)$ a nonzero element of $\Omega^\alpha_x M$ is given by

$$|\mathrm{d}x|^\alpha := |\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^n|^\alpha.$$

Thus in fact $|\mathrm{d}x|^\alpha$ is a nonvanishing section of $\Omega^\alpha(M)$ defined on U, and this gives a local trivialization of $\Omega^\alpha(M)$ on U.

Next suppose that $\Phi : V \subset M \to V' \subset \mathbb{R}^n$ is another coordinate chart for M such that $U \cap V \neq \emptyset$, and let $g = \Psi \circ \Phi^{-1}$ be the change of coordinates map between the two charts. Also suppose that the coordinates corresponding to Φ are labeled $(y^1, \dots, y^n)$. Then we have

$$dx^j|_{\Phi^{-1}(y)} = \frac{\partial g^j}{\partial y^k}(y)\, \mathrm{d}y^k|_{\Phi^{-1}(y)}$$

and so

$$|\mathrm{d}x|^\alpha = \left|\det\left(\frac{\partial g^j}{\partial y^k}(z)\right)\right|^\alpha |\mathrm{d}y|^\alpha$$

for all $z \in U \cap V$. This shows that the transition map from the local trivialization given by Ψ on U and that given by Φ on V is

$$a \mapsto a\left|\det\left(\frac{\partial g^j}{\partial y^k}(\Phi(z))\right)\right|^\alpha$$

for all $z \in U \cap V$.

Our main application of half densities will be in relation to studying the principal symbol of a Fourier integral operator. In this case we will need to consider the bundle of half densities over an immersed conic Lagrangian manifold as discussed in the previous sections of this chapter. Indeed, suppose that $\phi(x,\xi) \in X \times C^\infty(\mathbb{R}^N \setminus \{0\})$ is a nondegenerate phase function parametrizing the conic Lagrangian $\Lambda_\phi \subset X \times (\mathbb{R}^n \setminus \{0\})$. We now describe a construction through which ϕ defines a nonvanishing section of $\Omega^{1/2}(\Lambda_\phi)$ and therefore also a local trivialization for $\Omega^{1/2}(\Lambda_\phi)$. We start by defining, for any $(x,\xi) \in C_\phi$, an element d_{C_ϕ} of $\Lambda^n_{(x,\xi)}(C_\phi)$ by the formula

$$\begin{aligned} \mathrm{d}_{C_\phi}(v_1,\, \dots v_n)\, [\mathrm{d}(\partial_{\xi_1}\phi) \wedge\, \dots\, \wedge \mathrm{d}(\partial_{\xi_N}\phi)]\, (v_{n+1},\, \dots, v_{n+N}) \\ = \left[\mathrm{d}x^1 \wedge\, \dots\, \wedge \mathrm{d}x^n \wedge \mathrm{d}\xi_1 \wedge\, \dots\, \wedge \mathrm{d}\xi_N\right] (v_1,\, \dots, v_{n+N}) \end{aligned} \tag{4.5}$$

whenever $v_1, \dots, v_n \in T_{(x,\xi)}C_\phi$. That (4.5) gives a well-defined nonzero n-form d_{C_ϕ} follows from the assumption that ϕ is nondegenerate. Now we can transfer d_{C_ϕ} to an element $\mathrm{d}_\phi \in \Lambda^n_{(x,\xi)}\Lambda_\phi$ using a pullback by $(DT_\phi)^{-1}$:

$$\mathrm{d}_\phi = ((DT_\phi)^{-1})^* \mathrm{d}_{C_\phi}.$$

Finally, $|\mathrm{d}_\phi|^{1/2} \in \Omega^{1/2}_{T_\phi(x,\xi)}(\Lambda_\phi)$ and letting (x,ξ) vary over C_ϕ we define a nonvanishing section $|\mathrm{d}_\phi|^{1/2}$ of $\Omega^{1/2}(\Lambda_\phi)$.

Now suppose that $\phi(x,\xi)$ and $\tilde{\phi}(x,\tilde{\xi})$ are nondegenerate phase functions that both parametrize an open subset $U \subset \Lambda_\phi$. We would like to find the transition map between the local trivializations of $\Omega^{1/2}(\Lambda_\phi)$ over U corresponding to $|\mathrm{d}_\phi|^{1/2}$ and $|\mathrm{d}_{\tilde{\phi}}|^{1/2}$. Again we will focus on a particular point $(x_0,\eta_0) \in \Lambda_\phi$. For this, we must introduce an auxiliary function $\psi \in C^\infty(X)$ such that $\partial_x\psi(x_0) = \eta_0$ and the graph $\Lambda = \{(x, \partial_x\psi(x))\}$ is transverse to Λ_ϕ at (x_0,η_0). It is always possible to find such a function by choosing it for example to be the function given in (3.6). Also we will use the notation $T_\phi^{-1}(x_0,\eta_0) = (x_0,\xi_0)$ and $T_{\tilde{\phi}}^{-1}(x_0,\eta_0) = (x_0,\tilde{\xi}_0)$.

Since Λ and $F = \{(x_0,\eta) \in T^*X\}$ are transverse we have

$$T_{(x_0,\eta_0)}(T^*X) = T_{(x_0,\eta_0)}\Lambda \oplus T_{(x_0,\eta_0)}F$$

and so there is a well defined projection $p : T_{(x_0,\eta_0)}(T^*X) \to T_{(x_0,\eta_0)}F$ along $T_{(x_0,\eta_0)}\Lambda$. The composition of this projection with DT_ϕ is given explicitly by

$$p \circ DT_\phi(a^j\partial_{x^j} + b_j\partial_{\xi_j}) = \Big(a^j(\partial^2_{x^jx^k}\phi(x_0,\xi_0) - \partial^2_{x^jx^k}\psi(x_0)) + b_j\partial^2_{x^k\xi_j}\phi(x_0,\xi_0)\Big)\,\partial_{\eta_k}.$$

On the other hand we have a map $A_\phi : X \times \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N$ defined by

$$A_\phi(x,\xi) = \partial_\xi\phi(x,\xi)$$

and we can write the differential of A_ϕ at (x_0,ξ_0) explicitly as

$$DA_\phi(a^j\partial_{x^j} + b_j\partial_{\xi_j}) = \Big(a^j\partial^2_{x^j\xi_k}\phi(x_0,\xi_0) + b_j\partial^2_{\xi_j\xi_k}\phi(x_0,\xi_0)\Big)\,\partial_{\xi_k}.$$

We will compare d_{C_ϕ} and $(p \circ DT_\phi)^*\mathrm{d}\eta$. To accomplish this we use the same notation

$$Q = \begin{pmatrix} \partial^2_{xx}\phi(x_0,\xi_0) - \partial^2_{xx}\psi(x_0) & \partial^2_{x\xi}\phi(x_0,\xi_0) \\ \partial^2_{\xi x}\phi(x_0,\xi_0) & \partial^2_{\xi\xi}\phi(x_0,\xi_0) \end{pmatrix}$$

as in section 3.3. Combining the above calculations we for $v_1, \dots v_n \in \operatorname{Ker}(DA_\phi)$ and $v_{n+1}, \dots, v_{n+N} \in T_{(x_0,\xi_0)}(X \times (\mathbb{R}^N \setminus \{0\}))$ we have

$$\begin{aligned}
&(p \circ DT_\phi)^* \mathrm{d}\eta(v_1, \dots, v_n)\ (DA_\phi)^* \mathrm{d}\xi(v_{n+1}, \dots, v_{n+N}) \\
&\qquad = \mathrm{d}\eta(p \circ DT_\phi(v_1), \dots, p \circ DT_\phi(v_n))\, \mathrm{d}\xi(DA_\phi(v_{n+1}), \dots, DA_\phi(v_{n+N})) \\
&\qquad = \det(Q)(x_0, \xi_0) \mathrm{d}x \wedge \mathrm{d}\xi(v_1, \dots, v_{n+N}) \\
&\qquad = \det(Q)(x_0, \xi_0) \mathrm{d}_{C_\phi}(v_1, \dots, v_n)(DA_\phi)^* \mathrm{d}\xi(v_{n+1}, \dots, v_{n+N}).
\end{aligned}$$

Therefore

$$\det(Q)(x_0, \xi_0)\ \mathrm{d}_{C_\phi} = (p \circ DT_\phi)^* \mathrm{d}\eta.$$

Finally we may use this to show, writing $\tilde{Q}$ for the same matrix with ϕ replaced by $\tilde{\phi}$,

$$\begin{aligned}
\mathrm{d}_\phi &= ((DT_\phi)^{-1})^* \mathrm{d}C_\phi \\
&= \frac{1}{\det(Q)(x_0, \xi_0)} p^* \mathrm{d}\eta \\
&= \frac{\det(\tilde{Q})(x_0, \tilde{\xi}_0)}{\det(Q)(x_0, \xi_0)} \frac{1}{\det(\tilde{Q})(x_0, \tilde{\xi}_0)} p^* \mathrm{d}\eta \\
&= \frac{\det(\tilde{Q})(x_0, \tilde{\xi}_0)}{\det(Q)(x_0, \xi_0)} ((DT_{\tilde{\phi}})^{-1})^* \mathrm{d}C_{\tilde{\phi}} \\
&= \frac{\det(\tilde{Q})(x_0, \tilde{\xi}_0)}{\det(Q)(x_0, \xi_0)} \mathrm{d}_{\tilde{\phi}}.
\end{aligned}$$

We summarize the results of these calculations in the following lemma.

Lemma 4.8. *Suppose that $\phi \in C^\infty(X \times (\mathbb{R}^N \setminus \{0\}))$ and $\tilde{\phi} \in C^\infty(X \times (\mathbb{R}^{\tilde{N}} \setminus \{0\}))$ are two nondegenerate phase functions which each parametrize a Lagrangian manifold $\Lambda \in T^*X \setminus \{0\}$ in an open set $U \subset \Lambda$. If $|\mathrm{d}_\phi|^{1/2}$ and $|\mathrm{d}_{\tilde{\phi}}|^{1/2}$ are the corresponding local sections of $\Omega^{1/2}(\Lambda)$ defined on U then*

$$|\mathrm{d}_\phi|^{1/2} = \sqrt{\frac{\det(\tilde{Q})(T_{\tilde{\phi}}^{-1}(x, \eta))}{\det(Q)(T_\phi^{-1}(x, \eta))}} |\mathrm{d}_{\tilde{\phi}}|^{1/2}.$$

The transition map from the local trivialization of $\Omega^{1/2}(\Lambda)$ given by $|\mathrm{d}_\phi|^{1/2}$ and that given by $|\mathrm{d}_{\tilde{\phi}}|^{1/2}$ is thus

$$a \mapsto a \sqrt{\frac{\det(\tilde{Q})(T_{\tilde{\phi}}^{-1}(x, \eta))}{\det(Q)(T_\phi^{-1}(x, \eta))}}$$

In particular, let us consider the situation from section 4.2 in which $\Lambda \subset (Y \times X) \times (\mathbb{R}^{2m} \setminus \{0\})$ is the graph of a twisted canonical transformation (we are dropping the primes from Y and X). Suppose we have two different partitions I, J and I', J' as in lemma 4.6 so that both (y, x^I, ξ_J) and $(y, x^{I'}, \xi_{J'})$ provide local coordinates on some conic open set in Λ. Then the transformation from the local trivialization of $\Omega^{1/2}(\Lambda)$ given by $(y, x^{I'}, \xi_{J'})$ to the local trivialization given by (y, x^I, ξ_J) is multiplication by

$$\left|\frac{\partial(y, x^{I'}, \xi_{J'})}{\partial(y, x^I, \xi_J)}\right|^{1/2} = \left|\begin{pmatrix} \frac{\partial x^{I'\setminus I}}{\partial x^{I\setminus I'}} & \frac{\partial \xi_{J'\setminus J}}{\partial x^{I\setminus I'}} \\ \frac{\partial x^{I'\setminus I}}{\partial \xi_{J\setminus J'}} & \frac{\partial \xi_{J'\setminus J}}{\partial \xi_{J\setminus J'}} \end{pmatrix}\right|^{1/2}.$$